PERGAMON

International Journal of Solids and Structures 37 (2000) 7593–7608

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Time-dependent stress analysis in functionally graded materials

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Received 6 February 1999

Abstract

In this article, a joined cylinder with a functionally graded material (FGM) is considered. An analytical solution for the calculation of stresses in FGM is presented for the elastic and creep behavior of the materials. This analytical solution can be used to study the time and temperature dependence of the stresses in a structure with FGM.

Keywords: Analytical solution; Thermal loading; Material creep behavior; Joint with functionally graded material (FGM); Cylinder

1. Introduction

In many applications, ceramic–metal joints are used under high-temperature conditions. Here, the metal is used as a mechanical support structure and the ceramic as a coating for resisting the high temperature, for example, the thermal barrier coating system (TBCs). In many cases, the metals can creep. Therefore, the creep behavior should be considered in the stress analysis if the joints are used at high temperatures.

Due to the difference in the thermal expansion coefficients and in the elastic constants, high stresses develop after a homogeneous temperature change in such a ceramic–metal joint. For linear elastic material behavior, there even exist stress singularities at the intersection of the interface and the free edge. To reduce this stress level and to avoid the stress singularity, a functionally graded material (FGM) is usually introduced.

Stresses in FGM under thermal loading have been analyzed extensively with regard to the elastic material behavior in the past 10 years (Arai et al., 1990; Erdogan and Wu, 1993; Fukui and Yamanaka, 1993; Hirano and Teraki, 1993; Obata and Noda, 1994; Tanigawa, 1995; Yang, 1998a). However, there is a lack of the stress analysis considering material creep behavior. So far, there is no analytical solution for the calculation of the stresses depending on time, temperature and the transition functions of the material in FGM.

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PII: S0020-7683(99)00310-8

Nomenclature

T	temperature change
t	time
E	Young's modulus
v	Poisson's ratio
α	thermal expansion coefficient
r	coordinate in the radial direction
$\bar{r}$	given point in the creep layer
R_i, R_a	inner and outer radii of the cylinder
σ_{ij}	stress tensor
ϵ_{ij}	strain tensor
$\dot{\sigma}_{ij}$	rate of stress tensor
$\dot{\epsilon}_{ij}$	rate of strain tensor
S_{ij}	deviator stress tensor
σ_{eff}	effective stress
u	displacement in the r direction
$\dot{u}$	rate of the displacement in the r direction
D, N	creep material constants
n, A	parameters in the transition function of E
m, B	parameters in the transition function of αT
d	constant strain ϵ_z
$\dot{d}$	constant strain rate $\dot{\epsilon}_z$
x_1, x_2	exponent in the solution of u and $\dot{u}$
C_1, C_2	unknown constants in the solution of u for the case of $\epsilon_z = 0$
C_{1d}, C_{2d}	unknown constants in the solution of u for the case of $\epsilon_z = d$
D_1, D_2	unknown constants in the solution of $\dot{u}$ for the case of $\dot{\epsilon}_z = 0$
D_{1d}, D_{2d}	unknown constants in the solution of $\dot{u}$ for the case of $\dot{\epsilon}_z = \dot{d}$
F_0	resulting force of σ_z at the ends of a cylinder for the case of $\epsilon_z = 0$
F_d	resulting force of σ_z at the ends of a cylinder for the case of $\epsilon_z = d$

To obtain the basic results in most cases, in general, plates or cylinders are applied in the stress analysis. In a plate, the stress analysis is simpler than in a cylinder due to the geometry curvature. In this article, a joined cylinder with a graded material is considered. An analytical solution to calculate the stresses in FGM is presented considering the elastic and creep behavior of the materials. For the elastic material behavior, the solution is exact. For the creep material behavior, the solution is asymptotic. For the stress analysis after a longer time creeping, the iterative procedure is necessary and higher order asymptotic solutions have to be used. Using this analytical solution, it is easy to study the dependence of the stresses on time, temperature and the transition functions of the material in FGM. This solution can also be used easily for the stress distribution optimization in a joint with FGM, taking the creep behavior into account. For the stress distribution optimization, some mathematic methods have to be used, e.g. the gradient method, the swarm search method, etc. Examples for the stress distribution optimization, using these methods, considering material elastic behavior are given in Schaller et al. (1999). In the asymptotic solution, the relationship between the stresses or the displacements and the creep material data are known. Therefore, another application of this analytical solution is to determine the creep material data in FGM iteratively, if the displacement or the stresses in FGM can be measured experimentally.

2. Solutions for linear elastic behavior in FGM

For the stress analysis in a joint with FGM, having material creep behavior, the solutions of the stresses at a time equal to zero (i.e. the initial stress state) are needed, which correspond to the solution of materials with linear elastic behavior. In this section, equations to calculate such linear stresses in FGM analytically will be given briefly for three cases: (a) the strain in the z axis direction is zero (the coordinate system see Fig. 1), i.e. $\epsilon_z = 0$; (b) the strain in the z axis direction is a known constant, denoted as d; (c) the strain in the z axis direction is free, which corresponds to the situation in the center of a very long cylinder with no constraint in the z direction.

2.1. The case of ϵ_z being zero

For the case of a cylinder being subjected to a radial temperature change (i.e. $T = T(r)$), the stress distribution is axial symmetric. For the case of $\epsilon_z = 0$, the elastic stress–strain relations in each material, homogeneous or FGM, read

$$\epsilon_r = \frac{1}{E'}\left(\sigma_r - \frac{v}{1-v}\sigma_\theta\right) + (1+v)\alpha T, \tag{1}$$

$$\epsilon_\theta = \frac{1}{E'}\left(\sigma_\theta - \frac{v}{1-v}\sigma_r\right) + (1+v)\alpha T \tag{2}$$

with $E' = E/(1-v^2)$, or

$$\sigma_r = \frac{E(1-v)}{(1-2v)(1+v)}\left(\epsilon_r + \frac{v}{1-v}\epsilon_\theta - \frac{1+v}{1-v}\alpha T\right), \tag{3}$$

$$\sigma_\theta = \frac{E(1-v)}{(1-2v)(1+v)}\left(\epsilon_\theta + \frac{v}{1-v}\epsilon_r - \frac{1+v}{1-v}\alpha T\right), \tag{4}$$

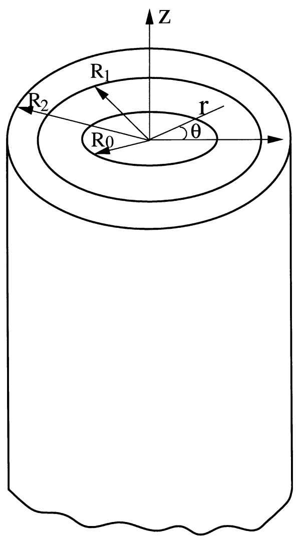

Fig. 1. The investigated geometry and the coordinate system.

$$\sigma_z = v(\sigma_r + \sigma_\theta) - E\alpha T, \tag{5}$$

where E is the Young's modulus, v, Poisson's ratio, α, the thermal expansion coefficient and T is the temperature distribution in the cylinder $T = T(r)$, here the temperature at a stress-free state is defined as zero. The strains are related to the displacement as

$$\epsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r}, \tag{6}$$

$$\epsilon_\theta = \frac{u}{r}, \tag{7}$$

where u is the displacement in the r-direction. The equilibrium equation for this problem is

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{8}$$

in each material. It should be noted that in FGM, the material data E, v and α are a function of the coordinates. In this article, it is assumed that $E = E(r)$, $\alpha = \alpha(r)$ and v is a constant, because the effect of v on the stresses is small. By inserting Eqs. (6) and (7) into Eqs. (3) and (4) and then into Eq. (8), the essential differential equation for the displacement u can be obtained as

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{\mathrm{d}u}{\mathrm{d}r}\left[\frac{1}{r} + \frac{\mathrm{d}(\ln(E))}{\mathrm{d}r}\right] + \frac{u}{r}\left[\frac{v}{1-v}\frac{\mathrm{d}(\ln(E))}{\mathrm{d}r} - \frac{1}{r}\right] = \frac{1+v}{1-v}\left[\frac{\mathrm{d}(\alpha T)}{\mathrm{d}r} + \alpha T\frac{\mathrm{d}(\ln(E))}{\mathrm{d}r}\right]. \tag{9}$$

When the solution of u is known from Eq. (9), the stresses and strains can be determined from Eqs. (3), (4), (6) and (7). Therefore, the important point is to find the analytical solution of Eq. (9). From Eq. (9), it can be seen that the solution of the problem is strongly dependent on the transition functions of E and α in the FGM and on the loading $T = T(r)$. Because the quantities α and T always appear together as a factor αT in the considered problem, in the following, αT will be dealt as one function. As an example, a power law transition function in the FGM and a power law function for the loading $T(r)$, i.e.

$$E = Ar^n, \quad \alpha T = Br^m; \tag{10}$$

will be discussed in this article. The procedure presented below is also valid for other forms of transition functions for E and αT. Through Eq. (10), the stress dependence on the coordinate and the temperature can be considered, because in Eq. (10), T may be a function of the coordinates.

The general solution of the displacement u for the given transition functions is

$$u(r) = C_1 r^{x_1} + C_2 r^{x_2} + \frac{v'' B(m+n)}{(m+1)(m+n+1) + (nv' - 1)} r^{(m+1)} \tag{11}$$

with

$$x_{1,2} = \frac{-n \pm \sqrt{n^2 + 4(1 - nv')}}{2},$$
$$v' = \frac{v}{1-v'}, \quad v'' = \frac{1+v}{1-v}. \tag{12}$$

The corresponding stresses are

$$\sigma_r(r) = \frac{(1-v)Ar^n}{(1-2v)(1+v)}\left\{C_1 r^{(x_1-1)}(v' + x_1) + C_2 r^{(x_2-1)}(v' + x_2) + \frac{Bm(v'-1)v'' r^m}{2m + m^2 + n + mn + nv'}\right\}, \tag{13}$$

$$\sigma_\theta(r) = \frac{(1-v)Ar^n}{(1-2v)(1+v)}\left\{C_1 r^{(x_1-1)}(1+v'x_1) + C_2 r^{(x_2-1)}(1+v'x_2) + \frac{Bm(1+n+m)(v'-1)v''r^m}{2m+m^2+n+mn+nv'}\right\}, \tag{14}$$

$$\sigma_z(r) = \frac{vA}{(1-2v)(1+v)}\left\{C_1 r^{n+x_1-1}(1+x_1) + C_2 r^{n+x_2-1}(1+x_2)\right\} - \frac{Av}{1+v}\frac{Bmv''(2+m+n)}{2m+m^2+mn+n+v'n}r^{m+n} - ABr^{m+n}. \tag{15}$$

To determine the unknown constants C_1 and C_2 in each material, boundary conditions have to be used, which are the continuity of the stress σ_r and displacement u at each interface and $\sigma_r = 0$ at the inner and outer surfaces.

For some special cases, the solutions can be simplified as follows:

1. The case of $n = 0$, but $m \neq 0$, means that the Young's modulus is a constant, but αT is a function of r, e.g. α is a constant and T is a function of r. The solution can be simplified as

$$x_1 = 1, \quad x_2 = -1, \tag{16}$$

$$u(r) = C_1 r + \frac{C_2}{r} + \frac{v''B}{m+2} r^{(m+1)}. \tag{17}$$

2. The case of $m = 0$, but $n \neq 0$, means that αT is a constant, but the Young's modulus is a function of r. The solution can be simplified as

$$u(r) = C_1 r^{x_1} + C_2^{x_2} + \frac{v''B}{1+v'} r, \tag{18}$$

where x_1 and x_2 depend on the value of n.

3. The case of $n = 0$ and $m = 0$, means that the Young's modulus is a constant and αT also is a constant, which corresponds to the case of a homogeneous material subjected to a homogeneous temperature change. The solution can be simplified as:

$$x_1 = 1, \quad x_2 = -1, \tag{19}$$

$$u(r) = C_1 r + \frac{C_2}{r} + \frac{B}{2}\frac{1+v}{1-v} r, \tag{20}$$

which is the same as that given in Yang (1998b) for multi-layer joined cylinder.

2.2. The case of ϵ_z being a known constant

For the case of ϵ_z being a known constant, denoted as d, the stress and strain relation is

$$\sigma_r = \frac{E(1-v)}{(1-2v)(1+v)}\left(\epsilon_r + \frac{v}{1-v}\epsilon_\theta + \frac{v}{1-v}d\right), \tag{21}$$

$$\sigma_\theta = \frac{E(1-v)}{(1-2v)(1+v)}\left(\epsilon_\theta + \frac{v}{1-v}\epsilon_r + \frac{v}{1-v}d\right), \tag{22}$$

$$\sigma_z = v(\sigma_r + \sigma_\theta) + Ed. \tag{23}$$

The relations between the displacement and strains, and the equilibrium equation for stresses are the same as those for the case of $\epsilon_z = 0$ (see Eqs. (6)–(8)).

The differential equation for displacement u is

$$\frac{d^2u}{dr^2} + \frac{du}{dr}\left[\frac{1}{r} + \frac{d(\ln(E))}{dr}\right] + \frac{u}{r}\left[\frac{v}{1-v}\frac{d(\ln(E))}{dr} - \frac{1}{r}\right] = -\frac{v}{1-v}\frac{d(\ln(E))}{dr}d. \tag{24}$$

For the transition functions given in Eq. (10), the solution of Eq. (24) is

$$u(r) = C_{1d}r^{x_1} + C_{2d}r^{x_2} - rvd. \tag{25}$$

The corresponding stresses are

$$\sigma_r(r) = \frac{(1-v)Ar^n}{(1-2v)(1+v)}\left\{C_{1d}r^{(x_1-1)}(v' + x_1) + C_{2d}r^{(x_2-1)}(v' + x_2)\right\}, \tag{26}$$

$$\sigma_\theta(r) = \frac{(1-v)Ar^n}{(1-2v)(1+v)}\left\{C_{1d}r^{(x_1-1)}(1 + v'x_1) + C_{2d}r^{(x_2-1)}(1 + v'x_2)\right\}, \tag{27}$$

$$\sigma_z(r) = \frac{vAr^n}{(1-2v)(1+v)}\left\{C_{1d}r^{(x_1-1)}(1 + x_1) + C_{2d}r^{(x_2-1)}(1 + x_2)\right\} + Ar^n d. \tag{28}$$

For the special case of $n = 0$, i.e. in a homogeneous material, the solutions are $x_1 = 1$, $x_2 = -1$ and

$$u = C_{1d}r + \frac{C_{2d}}{r}, \tag{29}$$

$$\sigma_r = \frac{E}{(1+v)(1-2v)}\left\{C_{1d} - \frac{(1-2v)C_{2d}}{r^2} + vd\right\}, \tag{30}$$

$$\sigma_\theta = \frac{E}{(1+v)(1-2v)}\left\{C_{1d} + \frac{(1-2v)C_{2d}}{r^2} + vd\right\}, \tag{31}$$

$$\sigma_z = \frac{E}{(1+v)(1-2v)}\{2C_{1d}v + (1-v)d\}, \tag{32}$$

which are the same as that given in Yang (1998b) for multi-layer joined cylinder. To determine the constants C_{1d} and C_{2d} in each material, boundary conditions have to be used.

2.3. The case of ϵ_z being free

In this section, the stress analysis in a cylinder with FGM under thermal loading and $\sigma_z = 0$ at the ends will be given. Finding an exact analytical solution satisfying $\sigma_z = 0$ at each point of the ends is very difficult. Although $\sigma_z = 0$ at the ends of the cylinder cannot be satisfied at each point, the solution of the resulting force at the ends equaling zero

$$2\pi\int_{R_i}^{R_a} \sigma_z r\,dr = 0 \tag{33}$$

is useful, where R_i is the inner radius and R_a is the outer radius of the cylinder (in Fig. 1, $R_i = R_0$ and $R_a = R_2$). Following the Saint–Venan principle, this solution can be used to calculate the stresses in the range far away from the ends of a cylinder.

The solution of Section 2.1 is for thermal loading, but for the case of $\epsilon_z = 0$. Under the assumption of $\epsilon_z = 0$, the resulting force of σ_z at the ends is

$$F_0 = \sum_{i=1}^{2} F_0^{(i)}, \tag{34}$$

where $F_0^{(i)}$ is the resulting force of σ_z in each material. The quantity $F_0^{(i)}$ can be calculated from

$$F_0^{(i)} = 2\pi \int_{R_i^{(i)}}^{R_a^{(i)}} \left\{ \frac{vA}{(1-2v)(1+v)} \left\{ C_1 r^{n+x_1-1}(1+x_1) + C_2 r^{n+x_2-1}(1+x_2) \right\} - \frac{Av}{1+v} \frac{Bmv''(2+m+n)}{2m+m^2+mn+n+v'n} r^{m+n} - ABr^{m+n} \right\} r \, \mathrm{d}r, \tag{35}$$

where $R_i^{(i)}$ and $R_a^{(i)}$ are the inner and outer radii of the ith layer, and the quantities A, B, C_1, C_2, x_1, x_2, m, n and v are applied for the ith material.

To equilibrate this force F_0 at the ends of the cylinder, a mechanical force denoted as F_d should be superposed, so that the resulting force of σ_z is zero at the ends of the cylinder. Under this force F_d, the strain ϵ_z in the range far away from the ends is a constant d. Therefore, the stresses can be calculated from the equations given in Section 2.2. The value of F_d can be obtained from

$$F_d = \sum_{i=1}^{2} F_d^{(i)} \tag{36}$$

with

$$F_d^{(i)} = 2\pi \int_{R_i^{(i)}}^{R_a^{(i)}} \left\{ \frac{vA}{(1-2v)(1+v)} \left\{ C_{1d} r^{(n+x_1-1)}(1+x_1) + C_{2d} r^{(n+x_2-1)}(1+x_2) \right\} + Ar^n d \right\} r \, \mathrm{d}r. \tag{37}$$

The coefficients C_{1d} and C_{2d} in each material are proportional to d. Therefore, $F_d^{(i)}$ can be rewritten as

$$F_d^{(i)} = 2\pi \mathrm{d} \int_{R_i^{(i)}}^{R_a^{(i)}} \left\{ \frac{vA}{(1-2v)(1+v)} \left\{ \tilde{C}_{1d} r^{(n+x_1-1)}(1+x_1) + \tilde{C}_{2d} r^{(n+x_2-1)}(1+x_2) \right\} + Ar^n \right\} r \, \mathrm{d}r, \tag{38}$$

with

$$\tilde{C}_{1d} = \frac{C_{1d}}{\mathrm{d}}, \quad \tilde{C}_{2d} = \frac{C_{2d}}{\mathrm{d}},$$

where $\tilde{C}_{1d}$ and $\tilde{C}_{2d}$ are independent of d. For the problem studied in this section, the value of d can be determined from the condition of

$$F_0 + F_d = 0. \tag{39}$$

Finally, under thermal loading, the stress distribution in a cylinder with FGM is

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^d, \tag{40}$$

where σ_{ij}^0 will be calculated from the equations in Section 2.1 and σ_{ij}^d will be determined from the equations in Section 2.2 using the value of d from Eq. (39) with Eqs. (34)–(36) and (38). Here, the equations are for a two-layer joint with FGM. Solutions for a joint with more layers with FGM can be found in Yang (1998b).

3. Solutions for creep behavior in FGM

Following Norton's law (Finnie and Heller, 1959), for materials with creep behavior, the relations between the rates of stress ($\dot{\sigma}_{ij}$) and strain ($\dot{\epsilon}_{ij}$) in the multi-axial form are:

$$\dot{\epsilon}_{ij} = \frac{1+v}{E}\dot{\sigma}_{ij} - \frac{v}{E}\dot{\sigma}_{kk}\delta_{ij} + \frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}S_{ij} \tag{41}$$

with

$$S_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}, \tag{42}$$

$$\sigma_{\text{eff}} = \sqrt{\tfrac{3}{2}S_{ij}S_{ij}} = \tfrac{1}{\sqrt{2}}\sqrt{(\sigma_r-\sigma_\theta)^2 + (\sigma_r-\sigma_z)^2 + (\sigma_z-\sigma_\theta)^2}, \tag{43}$$

where D and N are material constants for creep. For convenience, in the following, S_{rr}, $S_{\theta\theta}$ and S_{zz} will be replaced by S_r, S_θ and S_z, respectively.

The rates of strains and displacement satisfy

$$\dot{\epsilon}_r = \frac{\mathrm{d}\dot{u}}{\mathrm{d}r}, \quad \dot{\epsilon}_\theta = \frac{\dot{u}}{r}, \tag{44}$$

and the equilibrium equation of the stress rates is

$$\frac{\mathrm{d}\dot{\sigma}_r}{\mathrm{d}r} + \frac{\dot{\sigma}_r - \dot{\sigma}_\theta}{r} = 0. \tag{45}$$

For the case of $\dot{\epsilon}_z = 0$, the relations between stress rates and strain rates are

$$\dot{\sigma}_r = \frac{E(1-v)}{(1-2v)(1+v)}\left\{\dot{\epsilon}_r + \frac{v}{1-v}\dot{\epsilon}_\theta - \frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\left[S'_r + \frac{v}{1-v}S'_\theta\right]\right\}, \tag{46}$$

$$\dot{\sigma}_\theta = \frac{E(1-v)}{(1-2v)(1+v)}\left\{\dot{\epsilon}_\theta + \frac{v}{1-v}\dot{\epsilon}_r - \frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\left[S'_\theta + \frac{v}{1-v}S'_r\right]\right\}, \tag{47}$$

$$\dot{\sigma}_z = v(\dot{\sigma}_r + \dot{\sigma}_\theta) - \tfrac{3}{2}D\sigma_{\text{eff}}^{(N-1)}ES_z. \tag{48}$$

For the case of $\dot{\epsilon}_z$ being a constant at each time, denoted as $\dot{d}$, there is

$$\dot{\sigma}_r = \frac{E(1-v)}{(1-2v)(1+v)}\left\{\dot{\epsilon}_r + \frac{v}{1-v}\dot{\epsilon}_\theta - \frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\left[S'_r + \frac{v}{1-v}S'_\theta\right] + \frac{v}{1-v}\dot{d}\right\}, \tag{49}$$

$$\dot{\sigma}_\theta = \frac{E(1-v)}{(1-2v)(1+v)}\left\{\dot{\epsilon}_\theta + \frac{v}{1-v}\dot{\epsilon}_r - \frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\left[S'_\theta + \frac{v}{1-v}S'_r\right] + \frac{v}{1-v}\dot{d}\right\}. \tag{50}$$

$$\dot{\sigma}_z = E\dot{d} - \tfrac{3}{2}D\sigma_{\text{eff}}^{(N-1)}ES_z + v(\dot{\sigma}_r + \dot{\sigma}_\theta), \tag{51}$$

with

$$S'_r = S_r + vS_z, \qquad S'_\theta = S_\theta + vS_z. \tag{52}$$

It should be noted that in FGM all quantities E, v, D and N are functions of the coordinate r.

3.1. The case of $\dot{\epsilon}_z$ being zero

For the case of $\dot{\epsilon}_z = 0$, insertion of Eq. (44) into Eqs. (46) and (47) and then into Eq. (45) gives the differential equation for $\dot{u}$ in FGM

$$\frac{d^2\dot{u}}{dr^2}+\frac{d\dot{u}}{dr}\left[\frac{1}{r}+\frac{d(\ln(E))}{dr}\right]+\frac{\dot{u}}{r}\left[v'\frac{d(\ln(E))}{dr}-\frac{1}{r}\right]=\frac{d(ln(E))}{dr}\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}(S'_r+v'S'_\theta)$$
$$+\frac{d}{dr}\left[\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}(S'_r+v'S'_\theta)\right]+\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}(1-v')\frac{S'_r-S'_\theta}{r}. \tag{53}$$

When $\dot{u}$ is known, the stress rates can be calculated from Eqs. (46)–(48). Therefore, we focus on finding the analytical solution of Eq. (53) for FGM. From Eq. (53), it can be seen that the solution of $\dot{u}$ strongly depends on the transition functions of E, v, D and N. As an example, the transition functions of E and αT given in Eq. (10) are used, whereas the case of v, D and N being constant is studied in this article. The procedure given below for solving the problem is also valid for other transition functions for E and αT, and for D and N being functions of the coordinate r. Of course, the solution is more complicated.

In general, the quantities σ_{eff}, S'_r and S'_θ are very complicated functions of the coordinate r, even in an implicit function form. Therefore, it is almost impossible to find an exact analytical solution of Eq. (53), even for simple transition functions as given in Eq. (10). However, we can find an asymptotical solution of Eq. (53). At first, we assume that σ_{eff}, S'_r and S'_θ are constant, i.e. they are independent of the coordinate r. Then, the solution of Eq. (53) for FGM is

$$\dot{u}(r)=D_1r^{x_1}+D_2r^{x_2}+r\frac{\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\{S'_r(1+n-v')+S'_\theta(v'(n+1)-1)\}}{n(1+v')}, \tag{54}$$

where the unknown constants D_1 and D_2 in each material can be determined from the boundary conditions and the continuity requirements at the interface.

The corresponding stress rates are

$$\dot{\sigma}_r(r)=\frac{(1-v)Ar^n}{(1-2v)(1+v)}\left\{D_1r^{(x_1-1)}(v'+x_1)+D_2r^{(x_2-1)}(v'+x_2)+\frac{1-v'}{n}\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}(S'_r-S'_\theta)\right\}, \tag{55}$$

$$\dot{\sigma}_\theta(r)=\frac{(1-v)Ar^n}{(1-2v)(1+v)}\left\{D_1r^{(x_1-1)}(1+v'x_1)+D_2r^{(x_2-1)}(1+v'x_2)\right.$$
$$\left.+\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}(S'_r-S'_\theta)(1-v')\frac{1+n}{n}\right\}. \tag{56}$$

For a homogeneous material, i.e. $n=m=0$, the solutions are (for details see Yang (1998b))

$$\dot{u}=D_1r+\frac{D_2}{r}+\frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\left\{\left(S'_r+\frac{v}{1-v}S'_\theta\right)\left(r-\frac{R_i^2}{r}\right)\right.$$
$$\left.+\frac{1-2v}{1-v}(S'_r-S'_\theta)\left[r\ln(r)-\frac{r}{2}-\frac{R_i^2}{r}\ln(R_i)+\frac{R_i^2}{2r}\right]\right\}. \tag{57}$$

$$\dot{\sigma}_r=\frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\frac{E}{1-v^2}\left\{S'_r\left[\ln(r)+\frac{1}{2}\frac{R_i^2}{r^2}+(1-2v)\frac{R_i^2}{r^2}\ln(R_i)-\frac{1}{2}\right]\right.$$
$$\left.+S'_\theta\left[-\ln(r)+\frac{1}{2}\frac{R_i^2}{r^2}-(1-2v)\frac{R_i^2}{r^2}\ln(R_i)-\frac{1}{2}\right]\right\}+\frac{E}{(1-2v)(1+v)}\left[D_1-\frac{D_2}{r^2}(1-2v)\right], \tag{58}$$

$$\dot{\sigma}_\theta=\frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\frac{E}{1-v^2}\left\{S'_r\left[\ln(r)-\frac{1}{2}\frac{R_i^2}{r^2}-(1-2v)\frac{R_i^2}{r^2}\ln(R_i)+\frac{1}{2}\right]\right.$$
$$\left.+S'_\theta\left[-\ln(r)-\frac{1}{2}\frac{R_i^2}{r^2}+(1-2v)\frac{R_i^2}{r^2}\ln(R_i)-\frac{3}{2}\right]\right\}+\frac{E}{(1-2v)(1+v)}\left[D_1+\frac{D_2}{r^2}(1-2v)\right], \tag{59}$$

$$\dot{\sigma}_z = v\left\{\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\frac{E}{1-v^2}\left[(S'_r - S'_\theta)\ln(r) - S'_\theta\right] + \frac{2E}{(1-2v)(1+v)}D_1\right\} - \frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}ES_z, \tag{60}$$

where R_i is the inner radius of the creep layer.

When the stress rate is known, the calculation of stresses at any time t_i should be performed iteratively,

$$\sigma_{ij}^{(i)}(r,t_i) = \sigma_{ij}^{(i-1)}(r,t_{i-1}) + \dot{\sigma}_{ij}^{(i)}(r,t_i)\,\mathrm{d}t^{(i)}, \tag{61}$$

where

$$t_i = \sum_{k=0}^{i}\mathrm{d}t^{(k)}. \tag{62}$$

The solution of $t_i = 0$ corresponds to that for elastic material behavior. To calculate $\dot{\sigma}_{ij}^{(i)}(r,t_i)$, the stresses at the time t_{i-1} used.

This solution can be used for the following cases: (a) the gradient in FGM is not strong and (b) the thickness of the graded layer is not large. To obtain a generally useful solution, a higher order approximation of σ_{eff}, S'_r and S'_θ should be made. Because σ_{eff}, S'_r, S'_θ are complex functions of the coordinate r, Taylor's developments of the functions

$$\sigma_{\text{eff}}(r) = \sigma_{\text{eff}}(\bar{r}) + \frac{\frac{\mathrm{d}}{\mathrm{d}r}[\sigma_{\text{eff}}(r)]|_{r=\bar{r}}}{1!}(r-\bar{r}) + \frac{\frac{\mathrm{d}^2}{\mathrm{d}r^2}[\sigma_{\text{eff}}(r)]|_{r=\bar{r}}}{2!}(r-\bar{r})^2 + \frac{\frac{\mathrm{d}^3}{\mathrm{d}r^3}[\sigma_{\text{eff}}(r)]|_{r=\bar{r}}}{3!}(r-\bar{r})^3 + \cdots, \tag{63}$$

$$S'_r(r) = S'_r(\bar{r}) + \frac{\frac{\mathrm{d}}{\mathrm{d}r}[S'_r(r)]|_{r=\bar{r}}}{1!}(r-\bar{r}) + \frac{\frac{\mathrm{d}^2}{\mathrm{d}r^2}[S'_r(r)]|_{r=\bar{r}}}{2!}(r-\bar{r})^2 + \frac{\frac{\mathrm{d}^3}{\mathrm{d}r^3}[S'_r(r)]|_{r=\bar{r}}}{3!}(r-\bar{r})^3 + \cdots, \tag{64}$$

$$S'_\theta(r) = S'_\theta(\bar{r}) + \frac{\frac{\mathrm{d}}{\mathrm{d}r}[S'_\theta(r)]|_{r=\bar{r}}}{1!}(r-\bar{r}) + \frac{\frac{\mathrm{d}^2}{\mathrm{d}r^2}[S'_\theta(r)]|_{r=\bar{r}}}{2!}(r-\bar{r})^2 + \frac{\frac{\mathrm{d}^3}{\mathrm{d}r^3}[S'_\theta(r)]|_{r=\bar{r}}}{3!}(r-\bar{r})^3 + \cdots \tag{65}$$

have to be used, where $\bar{r}$ is one given point in the creep layer, e.g. the center point of the creep layer.

Then, the solution of $\dot{u}(r)$ has the following form:

$$\begin{aligned}\dot{u}(r) = {} & D_1 r^{x_1} + D_2 r^{x_2} + r\frac{\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}\left\{S'_r(1+n-v') + S'_\theta(v'(n+1)-1)\right\}}{n(1+v')} \\ & + U_1 r + U_2 r^2 + U_3 r^3 + \cdots,\end{aligned} \tag{66}$$

where the coefficients $U_1, U_2, U_3, \ldots$, are known and dependent on the values of $\mathrm{d}^p/\mathrm{d}r^p[\sigma_{\text{eff}}(r)]|_{r=\bar{r}}$, $\mathrm{d}^p/\mathrm{d}r^p[S'_r(r)]|_{r=\bar{r}}$, $\mathrm{d}^p/\mathrm{d}r^p[S'_\theta(r)]|_{r=\bar{r}}$, $p=0,1,2,3,\ldots$ The question of how many higher order terms should be used for the stress analysis is dependent on the gradient of the material properties in FGM. In Section 4, three examples will be given to show that the asymptotic solution is useful for the stress analysis in FGM, taking the material creep behavior into account.

3.2. The case of $\dot{\epsilon}_z$ being constant

For the case of $\dot{\epsilon}_z = \dot{d}$, where $\dot{d}$ may be a function of the time, the differential equation for $\dot{u}$ is

$$\begin{aligned}\frac{\mathrm{d}^2\dot{u}}{\mathrm{d}r^2} + \frac{\mathrm{d}\dot{u}}{\mathrm{d}r}\left[\frac{1}{r} + \frac{\mathrm{d}(\ln(E))}{\mathrm{d}r}\right] + \frac{\dot{u}}{r}\left[v'\frac{\mathrm{d}(\ln(E))}{\mathrm{d}r} - \frac{1}{r}\right] = {} & \frac{\mathrm{d}(\ln(E))}{\mathrm{d}r}\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}(S'_r + v'S'_\theta) \\ & + \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}(S'_r + v'S'_\theta)\right] + \frac{3}{2}D\sigma_{\text{eff}}^{(N-1)}(1-v')\frac{S'_r - S'_\theta}{r} - v'\dot{\mathrm{d}}\frac{\mathrm{d}(\ln(E))}{\mathrm{d}r}.\end{aligned} \tag{67}$$

In Eq. (67), only the term $-v'\dot{d}(\mathrm{d}(\ln(E))/\mathrm{d}r)$ is different from Eq. (53), which effects only on the special solution of $\dot{u}(r)$. The solution of $\dot{u}(r)$ with the assumption of $\sigma_{\mathrm{eff}},\ S'_r,\ S'_\theta$ being constant for FGM reads

$$\dot{u}(r) = D_{1d}r^{x_1} + D_{2d}r^{x_2} + r\frac{\frac{3}{2}D\sigma_{\mathrm{eff}}^{(N-1)}\{S'_r(1+n-v') + S'_\theta(v'(n+1)-1)\}}{n(1+v')} - rv\dot{d}. \tag{68}$$

The corresponding stresses are

$$\begin{aligned}\dot{\sigma}_r(r) = \frac{(1-v)Ar^n}{(1-2v)(1+v)}\Bigg\{&D_{1d}r^{(x_1-1)}(v'+x_1) + D_{2d}r^{(x_2-1)}(v'+x_2)\\ &+ \frac{1-v'}{n}\frac{3}{2}D\sigma_{\mathrm{eff}}^{(N-1)}(S'_r - S'_\theta)\Bigg\},\end{aligned} \tag{69}$$

$$\begin{aligned}\dot{\sigma}_\theta(r) = \frac{(1-v)Ar^n}{(1-2v)(1+v)}\Bigg\{&D_{1d}r^{(x_1-1)}(1+v'x_1) + D_{2d}r^{(x_2-1)}(1+v'x_2)\\ &+ \frac{3}{2}D\sigma_{\mathrm{eff}}^{(N-1)}(S'_r - S'_\theta)(1-v')\frac{1+n}{n}\Bigg\},\end{aligned} \tag{70}$$

$$\dot{\sigma}_z(r) = Ar^n\dot{\mathrm{d}} - \tfrac{3}{2}D\sigma_{\mathrm{eff}}^{(N-1)}Ar^nS_z + v(\dot{\sigma}_r(r) + \dot{\sigma}_\theta(r)). \tag{71}$$

For homogeneous material, the solution of $\dot{u}(r)$ is the same as that for the case of $\dot{\epsilon}_z = 0$ given in Eq. (57) (it should be noted that the values of D_1 and D_2 are different). The stresses are

$$\begin{aligned}\dot{\sigma}_r = \frac{1}{2}\times\frac{3}{2}D\sigma_{\mathrm{eff}}^{(N-1)}\frac{E}{1-v^2}\Bigg\{&S'_r\left[\ln(r) + \frac{1}{2}\frac{R_i^2}{r^2} + (1-2v)\frac{R_i^2}{r^2}\ln(R_i) - \frac{1}{2}\right]\\ &+ S'_\theta\left[-\ln(r) + \frac{1}{2}\frac{R_i^2}{r^2} - (1-2v)\frac{R_i^2}{r^2}\ln(R_i) - \frac{1}{2}\right]\Bigg\}\\ &+ \frac{E}{(1-2v)(1+v)}\left[D_1 - \frac{D_2}{r^2}(1-2v)\right] + \frac{Ev\dot{d}}{(1-2v)(1+v)},\end{aligned} \tag{72}$$

$$\begin{aligned}\dot{\sigma}_\theta = \frac{1}{2}\times\frac{3}{2}D\sigma_{\mathrm{eff}}^{(N-1)}\frac{E}{1-v^2}\Bigg\{&S'_r\left[\ln(r) - \frac{1}{2}\frac{R_i^2}{r^2} - (1-2v)\frac{R_i^2}{r^2}\ln(R_i) + \frac{1}{2}\right]\\ &+ S'_\theta\left[-\ln(r) - \frac{1}{2}\frac{R_i^2}{r^2} + (1-2v)\frac{R_i^2}{r^2}\ln(R_i) - \frac{3}{2}\right]\Bigg\}\\ &+ \frac{E}{(1-2v)(1+v)}\left[D_1 + \frac{D_2}{r^2}(1-2v)\right] + \frac{Ev\dot{d}}{(1-2v)(1+v)},\end{aligned} \tag{73}$$

$$\dot{\sigma}_z = E\dot{\mathrm{d}} - \tfrac{3}{2}D\sigma_{\mathrm{eff}}^{(N-1)}ES_z + v(\dot{\sigma}_r + \dot{\sigma}_\theta). \tag{74}$$

Eq. (67) differs from Eq. (53) only due to the term $-v'\dot{d}(\mathrm{d}(\ln(E))/\mathrm{d}r)$, which is a well known explicit function of r for a given $E = E(r)$. Therefore, the method to obtain the higher order asymptotic solution of $\dot{u}(r)$ is the same as that presented in Section 3.1.

For the stress analysis in a cylinder with FGM under thermal loading and $\sigma_z = 0$ at the ends, a similar procedure to that in Section 2.3 can be performed to determine the stress rates. The stress rates can be calculated from

$$\dot{\sigma}_{ij}(r) = \dot{\sigma}^0_{ij}(r) + \dot{\sigma}^d_{ij}(r), \tag{75}$$

where $\dot{\sigma}_{ij}^{0}$ is determined from the equations given in Section 3.1 and $\dot{\sigma}_{ij}^{d}$ from the equations given in Section 3.2. To determine the value of $\dot{d}$, the condition of

$$2\pi \int_{R_i}^{R_a} \dot{\sigma}_z(r) r \, \mathrm{d}r = 0 \tag{76}$$

is used. This solution can be used to calculate the stresses in the range far away from the ends of a cylinder with FGM.

4. Numerical results and discussion

In this section, three examples with different gradients in FGM will be presented to show that the described asymptotical solution is useful for the stress analysis in FGM with creep behavior. In all examples, the geometry, material data, transition function in FGM and the loading are fictive.

A two layer cylinder is considered, in which the inner layer is a homogeneous material with elastic behavior and the outer layer is an FGM with creep material behavior. The radii of the cylinder are $R_0 = 20$ mm, $R_1 = 41$ mm, $R_2 = 43.55$ mm. The data of material 1 are

$$E_1 = 215 \text{ GPa}, \quad v_1 = 0.3, \quad \alpha_1 = 16.28 \times 10^{-6} \text{ K}^{-1}, \tag{77}$$

and of material 2

$$E_2 = Ar^n \text{ GPa}, \quad v_2 = 0.3, \quad \alpha_2 T = Br^m, \tag{78}$$

$$D = 1.4 \times 10^{-8}, \quad N = 2.25, \tag{79}$$

where for D, the stresses should be given in MPa and the time in hours. The thermal loading is as follows: the initial temperature is 0°C, at which the stresses are free, and then t hours of creeping at 980°C. For all examples, there is

$$A = E_1/R_1^n, \quad B = \alpha_1 T/R_1^m. \tag{80}$$

For example 1, $n = -15$ and $m = 8$, such that at the outer surface $(r = R_2)$ $E_2 = 87$ GPa, $\alpha_2 = 26.38 \times 10^{-6}\,\text{K}^{-1}$ (gradient in FGM is stronger with $x_1 = 15.48$ and $x_2 = -0.4799$).

For example 2, $n = 2$ and $m = 3$, and at $r = R_2$ there is $E_2 = 242.6$ GPa, $\alpha_2 = 19.51 \times 10^{-6}\,\text{K}^{-1}$ (gradient in FGM is weaker with $x_1 = 0.6904$ and $x_2 = -2.069$).

For example 3, $n = 15$ and $m = -8$ and at $r = R_2$ there is $E_2 = 531.5$ GPa, $\alpha_2 = 10.05 \times 10^{-6}\,\text{K}^{-1}$ (gradient in FGM is stronger with $x_1 = -0.3711$ and $x_2 = -14.63$).

The results presented below are the stresses in the range far away from the ends of a cylinder with FGM, and σ_z being zero at the ends.

For comparison, the stresses calculated from the asymptotical solution with different higher order approximation and from the finite element method (FEM) are shown in the figures. The FE calculations are performed by using the program ABAQUS (ABAQUS, 1994). In FE calculations, for different transition function in FGM the convergence speed of the solution is different. Therefore, for the same used CPU time, the obtained results correspond to different creeping times and the creeping times have no real meaning.

For example 1, the stress distributions after 5 h of creeping are plotted in Fig. 2(a)–(c) for the stress components σ_r, σ_θ and σ_z, respectively. It can be seen that for this transition functions ($n = -15$ and $m = 8$), the second-order asymptotic solution is sufficient for the calculation of the stresses.

For example 2, the stress distributions after 4.94 h of creeping are plotted in Fig. 3(a)–(c) for the stress components σ_r, σ_θ and σ_z, respectively. It is obvious that for this transition functions ($n = 2$ and $m = 3$), the first-order asymptotic solution can be used approximately for the calculation of the stresses, but with a higher order solution the agreement of the stresses from the asymptotic solution and FEM is better.

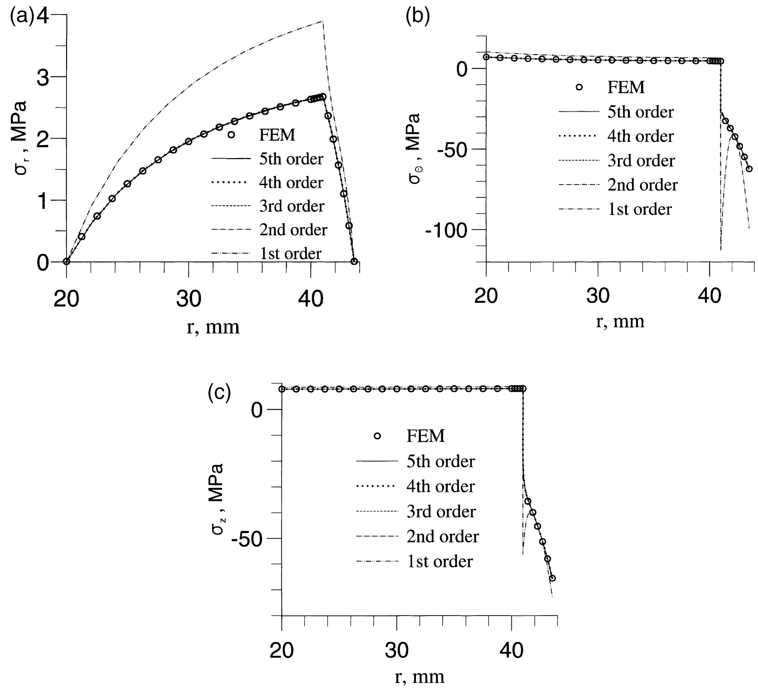

Fig. 2. Comparison of the stresses calculated from the asymptotic solution and FEM for example 1 (5 h of creeping, $n = -15$ and $m = 8$).

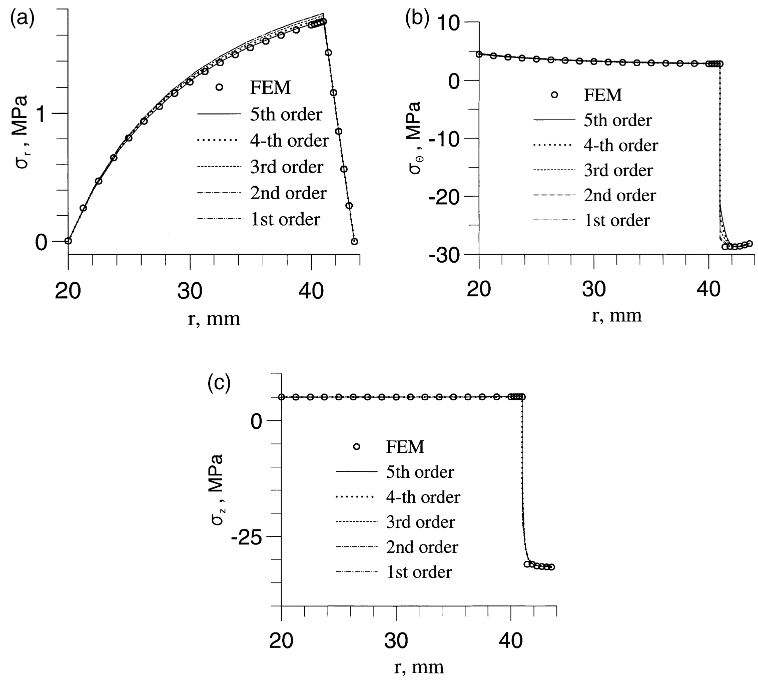

Fig. 3. Comparison of the stresses calculated from the asymptotic solution and FEM for example 2 (4.94 h of creeping, $n = 2$ and $m = 3$).

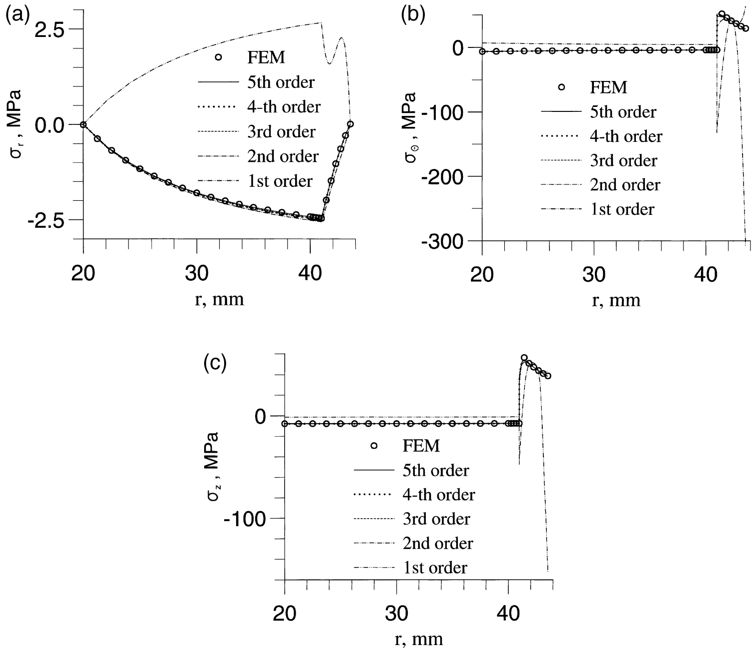

Fig. 4. Comparison of the stresses calculated from the asymptotic solution and FEM for example 3 and for long time creeping (2.22 h of creeping, $n=15$ and $m=-8$).

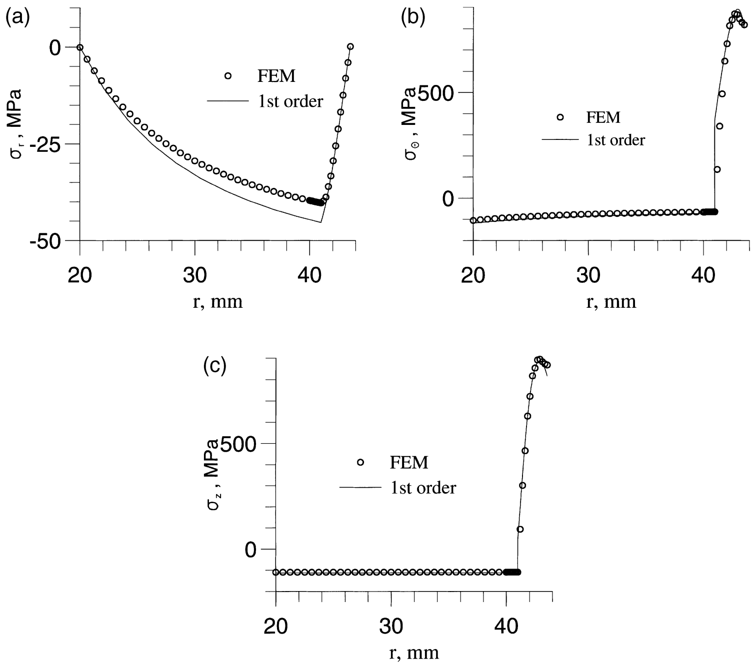

Fig. 5. Comparison of the stresses calculated from the asymptotic solution and FEM for example 3 and for short time creeping (0.0309 h of creeping, $n=15$ and $m=-8$).

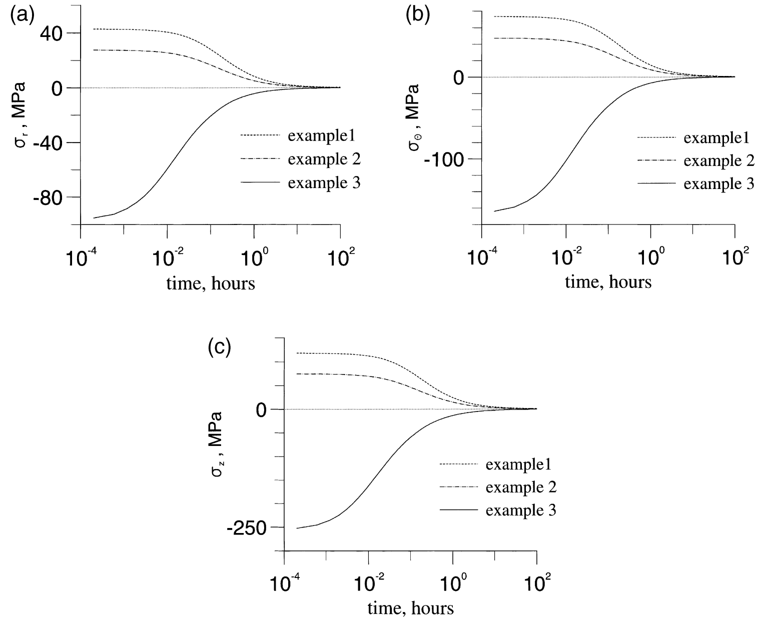

Fig. 6. Time-dependent stresses at the point in elastic material for all the three examples.

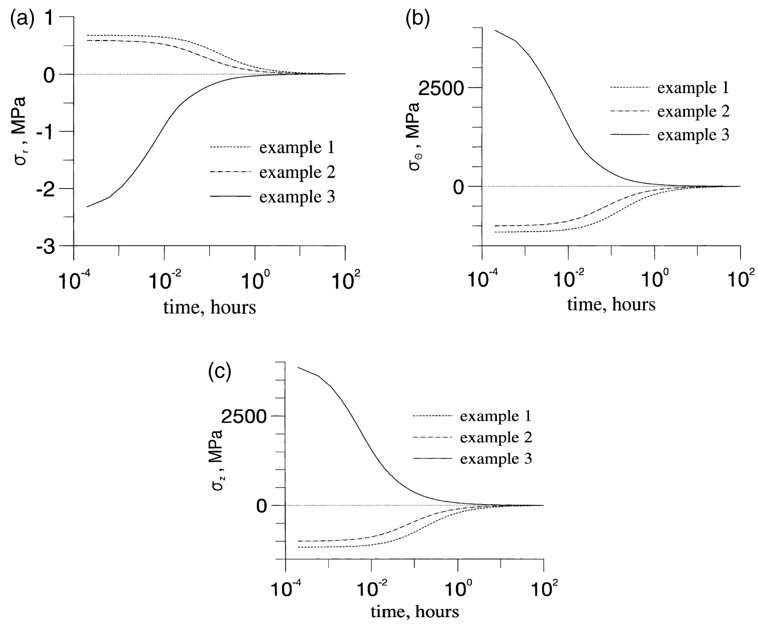

Fig. 7. Time-dependent stresses at the point in creep material for all the three examples.

For example 3, the stress distributions after 2.22 h of creeping are plotted in Fig. 4(a)–(c) for the stress components σ_r, σ_θ and σ_z, respectively. It can be seen that for this transition functions ($n = 15$ and $m = -8$), the first- and second-order asymptotic solution cannot be used for the calculation of the stresses (see Fig. 4 b), only the results from the third or higher order asymptotic solution agree well with those of FEM. However, for a short time creeping, the first order asymptotic solution can be used to calculate the stresses approximately. Fig. 5(a)–(c) show the stress distribution after 0.0309 h of creeping for example 3.

A lot of FEM calculations have shown that the fifth-order asymptotic solution can be used well to calculate the stresses for a long time creeping.

Using the fifth-order asymptotic solution, we can study the stress' time dependence. The time-dependent stresses at given points for the three examples is plotted in Fig. 6 for the point $r = 38.9$ mm (in material 1) and in Fig. 7 for the point $r = 42.35$ mm (in material 2). It can be seen that after 10 h of creeping, all stresses are reduced to almost zero. It should be noted that this is only true in a two layer joint with creep material behavior. For a three or more layer joint with creep material behavior, the stresses are not reduced to zero, but to a constant (Yang, 1998b).

5. Conclusions

In a cylindrical joint, the analytical solutions to calculate thermal stresses in FGM are found for the material with elastic and creep behavior. For the elastic material behavior, the solution is exact. For the material with creep, the solution is asymptotic. A lot of FEM calculations have shown that the fifth-order asymptotic solution can be used to calculate the stresses for a long time creeping. However, for a short time creeping, the first order asymptotic solution is enough to calculate the stresses approximately.

This analytical solution can be used easily to study the dependence of the stresses on time, temperature and the transition functions of the material in FGM. The solution can also be used for the stress distribution optimization in a joint with FGM taking the creep behavior into account.

Acknowledgements

The financial support of the Deutsche Forschungsgemeinschaft is gratefully acknowledged. The author would like to thank Prof. Munz and Dr. Fett for their useful discussions.

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